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Bäcklund transformation on surfaces with $K = -1$ in $\mathbb{R}^{2,1}$ ★

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Abstract

In this paper, we generalize the Bäcklund theorem on surfaces with Gaussian curvature $K = -1$ in $\mathbb{R}^3$ to the surfaces with $K = -1$ in $\mathbb{R}^{2,1}$.

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The famous Bäcklund theorem presented a geometrical method to construct a family of surfaces with Gaussian curvature $K = -1$ from a known surface with $K = -1$, i.e., the Bäcklund transformation that we know well [1–3]. With the research and development of the soliton theory, Bäcklund transformation has become an important method to find the solutions of soliton equations. At the same time, the geometricians also pay attention to the generalization and development of the geometrical content of the Bäcklund theorem. In [4,5], the authors generalized the Bäcklund theorem to the n-dimensional submanifolds with negative constant curvature in E^{2n-1}. In [6], we generalize the Bäcklund theorem to the surfaces with $(k_1 - m)(k_2 - m) = -l^2$ in $\mathbb{R}^3$, where k_1 and k_2 are the principal curvatures. In this paper, we generalize the Bäcklund theorem to the surfaces with $K = -1$ in Minkowski space $\mathbb{R}^{2,1}$.

It is known [7], that for the surfaces with $K = -1$ in $\mathbb{R}^{2,1}$, we have the following propositions.

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Proposition 1. *If S is a space-like surface of $K = -1$ in $\mathbb{R}^{2,1}$ and free of umbilics, then S can be covered by charts with Tchebyshev coordinates (u, v) such that the first fundamental form*

$$\mathrm{I} = \cosh^2 \tfrac{1}{2}\alpha \, \mathrm{d}u^2 + \sinh^2 \tfrac{1}{2}\alpha \, \mathrm{d}v^2, \tag{1}$$

and the second fundamental form

$$\mathrm{II} = \cosh \tfrac{1}{2}\alpha \sinh \tfrac{1}{2}\alpha (\mathrm{d}u^2 + \mathrm{d}v^2), \tag{2}$$

where $\alpha(u, v)$ satisfies the sinh-*Laplace equation*

$$\alpha_{uu} + \alpha_{vv} = \sinh \alpha. \tag{3}$$

Proposition 2. *If S' is a time-like surface with $K = -1$ in $\mathbb{R}^{2,1}$ and free of umbilics, then S' can be covered by charts with Tchebyshev coordinate (u, v) such that the first fundamental form*

$$\mathrm{I} = \cos^2 \tfrac{1}{2}\alpha' \, \mathrm{d}u^2 - \sin^2 \tfrac{1}{2}\alpha' \, \mathrm{d}v^2, \tag{4}$$

and the second fundamental form

$$\mathrm{II} = \cos \tfrac{1}{2}\alpha' \sin \tfrac{1}{2}\alpha' (\mathrm{d}u^2 + \mathrm{d}v^2), \tag{5}$$

where α' satisfies the sine-Laplace equation

$$\alpha'_{uu} + \alpha'_{vv} = \sin \alpha'. \tag{6}$$

Theorem 1. *If $\alpha(u, v)$ satisfies Eq.* (3), *τ is an arbitrary constant, then the following equations on α':*

$$\tfrac{1}{2} \cosh \tau (\alpha'_v + \alpha_u) = -\sinh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha' - \sinh \tau \cosh \tfrac{1}{2}\alpha \sin \alpha', \tag{7}$$

$$\tfrac{1}{2} \cosh \tau (\alpha'_u - \alpha_v) = \cosh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha' - \sinh \tau \sinh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha' \tag{8}$$

are completely integrable, and $\alpha'(u, v)$ satisfies the equation:

$$\alpha'_{uu} + \alpha'_{vv} = \sin \alpha'. \tag{9}$$

Proof. Since (7) and (8),

$$\begin{aligned} \tfrac{1}{2} \cosh \tau (\alpha'_{vu} + \alpha_{uu}) = {} & -\tfrac{1}{2}\alpha_u (\cosh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha' + \sinh \tau \sinh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha') \\ & + \tfrac{1}{2}\alpha'_u (\sinh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha' - \sinh \tau \cosh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha'), \\ \tfrac{1}{2} \cosh \tau (\alpha'_{uv} - \alpha_{vv} = {} & \tfrac{1}{2}\alpha_v (\sinh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha' - \sinh \tau \cosh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha') \\ & + \tfrac{1}{2}\alpha'_v (\cosh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha' + \sinh \tau \sinh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha'), \end{aligned}$$

then

$$\begin{aligned}&\tfrac{1}{2}\cosh\tau(\alpha'_{vu}-\alpha'_{uv}+\alpha_{uu}+\alpha_{vv})\\&\quad=-\tfrac{1}{2}(\alpha_u+\alpha'_v)(\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha'+\sinh\tau\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha')\\&\qquad+\tfrac{1}{2}(\alpha'_u-\alpha_v)(\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha'-\sinh\tau\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha').\end{aligned}\tag{10}$$

Substituting (7) and (8) into (10), we have

$$\begin{aligned}&\tfrac{1}{2}\cosh^2\tau(\alpha'_{vu}-\alpha'_{uv}+\alpha_{uu}+\alpha_{vv})\\&\quad=(\sinh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha'+\sinh\tau\cosh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha')\\&\qquad\times(\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha'+\sinh\tau\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha')\\&\qquad+(\cosh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha'-\sinh\tau\sinh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha')\\&\qquad\times(\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha'-\sinh\tau\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha')\\&\quad=\tfrac{1}{2}\cosh^2\tau\sinh\alpha.\end{aligned}$$

Since α satisfies (3), (7) and (8) are completely integrable. Since (7) and (8),

$$\begin{aligned}\tfrac{1}{2}\cosh\tau(\alpha'_{vv}+\alpha_{uv})&=-\tfrac{1}{2}\alpha_v(\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha'+\sinh\tau\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha')\\&\quad+\alpha'_v(\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha'-\sinh\tau\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha'),\\\tfrac{1}{2}\cosh\tau(\alpha'_{uu}-\alpha_{vu})&=\tfrac{1}{2}\alpha_u(\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha'-\sinh\tau\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha')\\&\quad+\alpha'_u(\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha'+\sinh\tau\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha'),\end{aligned}$$

then

$$\begin{aligned}\tfrac{1}{2}\cosh\tau(\alpha'_{uu}+\alpha_{vv})&=\tfrac{1}{2}(\alpha'_u-\alpha_v)(\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha'+\sinh\tau\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha')\\&\quad+\tfrac{1}{2}(\alpha'_v+\alpha_u)(\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha'-\sinh\tau\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha').\end{aligned}\tag{11}$$

Substituting (7) and (8) into (11), we have

$$\alpha'_{uu}+\alpha'_{vv}=\sin\alpha'.$$

The theorem is proved. □

By a similar proof, we also have the following theorem.

Theorem 1A. *If α' satisfies Eq.* (6), *τ is an arbitrary constant then Eqs.* (7) *and* (8) *on α are completely integrable, and $\alpha(u, v)$ satisfies Eq.* (3).

Therefore, (7) and (8) give the Bäcklund transformation between (3) and (6).

In particular, when $\tau = 0$, this Bäcklund transformation was mentioned in [8].

Suppose S is a space-like surface with $K = -1$ covered by Tchebyshev coordinate (u, v), $r(u, v)$ is a parameter representation of S with I and II as (1) and (2). Let (r, e_1, e_2, e_3) be a field of orthonormal frames such that e_1 and e_2 are the unit tangent vectors of u-lines, and v-lines, respectively, e_3 is the normal vector of $S(e_1^2 = e_2^2 = -e_3^2 = 1)$, then we have the moving equations:

$$\mathrm{d}r = w_1 e_1 + w_2 e_2, \qquad \mathrm{d}e_i = \sum_{j=1}^{3} w_{ij} e_j, \quad i = 1, 2, 3,$$

where

$$w_1 = \cosh \tfrac{1}{2}\alpha \, \mathrm{d}u, \qquad w_2 = \sinh \tfrac{1}{2}\alpha \, \mathrm{d}v, \tag{12}$$

$$w_{12} = -w_{21} = -\tfrac{1}{2}\alpha_v \, \mathrm{d}u + \tfrac{1}{2}\alpha_u \, \mathrm{d}v, \tag{13}$$

$$w_{13} = w_{31} = \sinh \tfrac{1}{2}\alpha \, \mathrm{d}u, \qquad w_{23} = w_{32} = \cosh \tfrac{1}{2}\alpha \, \mathrm{d}v. \tag{14}$$

Let

$$e = \cos \tfrac{1}{2}\alpha' e_1 + \sin \tfrac{1}{2}\alpha' e_2, \qquad e^{\perp} = -\sin \tfrac{1}{2}\alpha' e_1 + \cos \tfrac{1}{2}\alpha' e_2,$$

$$e_3' = \cosh \tau e^{\perp} - \sinh \tau e_3,$$

where α' is a solution of Eqs. (7) and (8).
Suppose S' is a surface defined by

$$r' = r + \cosh \tau e.$$

Theorem 2. *e_3' is the normal vector of S'.*

Proof. Since

$$\mathrm{d}r' = \mathrm{d}r + \cosh \tau \, \mathrm{d}e, \tag{15}$$

$$\mathrm{d}e = \tfrac{1}{2}(\mathrm{d}\alpha' + 2w_{12})e^{\perp} + (\cos \tfrac{1}{2}\alpha w_{13} + \sin \tfrac{1}{2}\alpha w_{23})e_3, \tag{16}$$

$$\begin{aligned} e_3' \cdot \mathrm{d}r' = {} & \cosh \tau ((\tfrac{1}{2} \cosh \tau (\alpha_u' - \alpha_v) - \cosh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha' \\ & + \sinh \tau \sinh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha') \, \mathrm{d}u + (\tfrac{1}{2} \cosh \tau (\alpha_v + \alpha_u) \\ & + \sinh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha' + \sinh \tau \cosh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha') \, \mathrm{d}v), \end{aligned} \tag{17}$$

since (7) and (8),

$$e_3' \cdot \mathrm{d}r' = 0,$$

and

$$e_3' \cdot e_3' = \cosh^2 \tau - \sinh^2 \tau = 1.$$

The theorem is proved. □

Lemma 1. *For the surface S', the first fundamental form*

$$\mathrm{I} = \cos^2 \tfrac{1}{2}\alpha' \, \mathrm{d}u^2 - \sin^2 \tfrac{1}{2}\alpha' \, \mathrm{d}v^2. \tag{18}$$

Proof. Since (15), (16) and

$$(\mathrm{d}\alpha' + 2w_{12}) = (\alpha'_u - \alpha_v)\,\mathrm{d}u + (\alpha'_v + \alpha_u)\,\mathrm{d}v,$$

and by using (7) and (8), then

$$\begin{aligned}
\mathrm{I} &= \mathrm{d}r' \cdot \mathrm{d}r' \\
&= (\cosh \tfrac{1}{2}\alpha \, \mathrm{d}u - \sin \tfrac{1}{2}\alpha'((\cosh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha' - \sinh \tau \sinh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha')\,\mathrm{d}u \\
&\quad - (\sinh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha' - \sinh \tau \cosh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha')\,\mathrm{d}v))^2 \\
&\quad + (\sinh \tfrac{1}{2}\alpha + \cos \tfrac{1}{2}\alpha'((\cosh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha' - \sinh \tau \sinh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha')\,\mathrm{d}u \\
&\quad - (\sinh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha' - \sinh \tau \cosh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha')\,\mathrm{d}v))^2 \\
&\quad - \cosh^2 \tau (\cos \tfrac{1}{2}\alpha' \sinh \tfrac{1}{2}\alpha \, \mathrm{d}u + \sin \tfrac{1}{2}\alpha' \cosh \tfrac{1}{2}\alpha \, \mathrm{d}v)^2 \\
&= \cos^2 \tfrac{1}{2}\alpha' \, \mathrm{d}u^2 - \sin^2 \tfrac{1}{2}\alpha' \, \mathrm{d}v^2.
\end{aligned}$$

This completes the proof. □

Lemma 2. *The second fundamental form of S'*

$$\mathrm{II} = \sin \tfrac{1}{2}\alpha' \cos \tfrac{1}{2}\alpha'(\mathrm{d}u^2 + \mathrm{d}v^2). \tag{19}$$

Proof. Since (15) and

$$\begin{aligned}
\mathrm{d}e'_3 &= \cosh \tau \, \mathrm{d}e^{\perp} - \sinh \tau \, \mathrm{d}e_3, \\
\mathrm{d}e^{\perp} &= -\tfrac{1}{2}(\mathrm{d}\alpha' + 2w_{12})e + (-\sin \tfrac{1}{2}\alpha w_{13} + \cos \tfrac{1}{2}\alpha' w_{23})e_3,
\end{aligned}$$

we have

$$\begin{aligned}
\mathrm{II} &= -\,\mathrm{d}R' \cdot \mathrm{d}e'_3 \\
&= \sinh \tau \cosh \tfrac{1}{2}\alpha \sinh \tfrac{1}{2}\alpha(\mathrm{d}u^2 + \mathrm{d}v^2) \\
&\quad + \tfrac{1}{2} \cosh \tau (\mathrm{d}\alpha'_2 + w_{12}) \\
&\quad \times ((\cosh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha' - \sinh \tau \sinh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha')\,\mathrm{d}u \\
&\quad + (\sinh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha' + \sinh \tau \cosh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha')\,\mathrm{d}v) \\
&\quad + \cosh^2 \tau (\cos \tfrac{1}{2}\alpha' \sinh \tfrac{1}{2}\alpha \, \mathrm{d}u + \sin \tfrac{1}{2}\alpha' \cosh \tfrac{1}{2}\alpha \, \mathrm{d}v) \\
&\quad \times (-\sinh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\alpha' \, \mathrm{d}u + \cosh \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha' \, \mathrm{d}v),
\end{aligned}$$

by using (7) and (8),

$$\mathrm{II} = \cos \tfrac{1}{2}\alpha' \sin \tfrac{1}{2}\alpha'(\mathrm{d}u^2 + \mathrm{d}v^2).$$

Then we have the following theorem.

Theorem 3. *S' is a time-like surface with $K = -1$.*

According to Theorems 1–3, to construct a family of time-like surfaces with $K = -1$ from a known space-like surface with $K = -1$, we only need to solve the completely integrable equations (7) and (8)

Inversely, suppose S' is a time-like surface with $k = -1$ covered by Tchebyshev coordinate (u, v), $r'(u, v)$ is a parameter representation of S' with I and II as (4) and (5). Let (r', e'_1, e'_2, e'_3) be a field of orthonormal frames such that e'_1 and e'_2 are the tangent vectors of u-lines and v-lines, respectively, e'_3 is the normal vector of $S'(e'^2_1 = -e'^2_2 = e'^2_3 = 1)$, then we have the moving equations:

$$\mathrm{d}r' = w'_1 e'_1 + w'_2 e'_2, \qquad \mathrm{d}e'_i = \sum_{j=1}^{3} w'_{ij} e'_j, \quad i = 1, 2, 3,$$

where

$$\begin{aligned} &w'_1 = \cos\tfrac{1}{2}\alpha'\,\mathrm{d}u, \quad w'_2 = \sin\tfrac{1}{2}\alpha'\,\mathrm{d}v, \quad w'_{12} = w'_{21} = -\tfrac{1}{2}\alpha'_v\,\mathrm{d}u + \tfrac{1}{2}\alpha'_u\,\mathrm{d}v, \qquad \square \\ &w'_{13} = -w'_{31} = \sin\tfrac{1}{2}\alpha,\ \mathrm{d}u, \qquad w'{}_{23} = w'_{32} = \cos\tfrac{1}{2}\alpha'\,\mathrm{d}v. \end{aligned}$$

Let

$$\begin{aligned} &e' = \cosh\tfrac{1}{2}\alpha e'_1 - \sinh\tfrac{1}{2}\alpha e'_2, \quad e'^{\perp} = \sinh\tfrac{1}{2}\alpha e'_1 - \cosh\tfrac{1}{2}\alpha e'_2, \\ &e_3 = \sinh\tau e'_3 - \cosh\tau e'^{\perp}, \end{aligned}$$

where $\alpha(u, v)$ is a solution of Eqs. (7) and (8).

Suppose S is defined by

$$r = r' - \cosh\tau e'.$$

For the surface S, we can also prove the following results.

Theorem 2A. *e_3 is a unit normal vector of S.*

Theorem 3A. *S is a space-like surface with $K = -1$.*

In conclusion, we give the Bäcklund transformation between the space-like surface with $K = -1$ and the time-like surface with $K = -1$ in $\mathbb{R}^{2,1}$.

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